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# Bridging the gap between formal structure and cognitive representation: A systematic review of metric space topology learning

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**Abstract**: Metric space topology is a foundational yet challenging topic in undergraduate mathematics. This systematic literature review examines how students learn metric space topology by synthesizing research on cognitive processes, learning obstacles, and instructional approaches. Following PRISMA 2020 guidelines, we searched Scopus, Web of Science, and Google Scholar for publications from 2015-2024. Two reviewers independently screened titles, abstracts, and full texts. The final analysis included 38 peer-reviewed articles. Data were extracted using a standardized framework and analyzed through thematic analysis. Four major learning obstacles emerged: challenges in translating intuitive understanding to formal definitions; difficulties in treating mathematical operations as formal objects (within APOS theory); gaps between mathematical terminology and student reasoning (commognitive perspective); and challenges in understanding and constructing proofs. Analysis revealed five phases in typical cognitive development: (1) applying procedures from calculus, (2) becoming aware of underlying structures, (3) developing topological reasoning, (4) integrating formal definitions with examples, and (5) abstract thinking with proof-based reasoning. Findings suggest that instruction may benefit from: connecting formal mathematics to students' existing understanding, providing scaffolded proof instruction, and explicitly developing mathematical language and concepts. These insights may inform course design and pedagogical approaches in advanced mathematics.
**Keywords**: advanced mathematics; metric space topology; learning difficulties; cognitive development; proof comprehension

## INTRODUCTION

Advanced analysis is a foundational course in higher mathematics that introduces formal, proof-based reasoning. Unlike earlier mathematics courses, advanced analysis requires students to move from intuitive, calculation-focused thinking to rigorous reasoning about mathematical structures and their properties. Many researchers identify this transition as particularly challenging in undergraduate mathematics education.

Research documents that students encounter significant difficulties when learning metric space topology. García & Flores (2021) found that approximately 75% of mathematics

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students in their study experienced difficulties when entering metric space topology. These findings suggest this is a widespread challenge rather than an isolated problem. The difficulties students experience appear to involve more than computational skills; they seem rooted in conceptual understanding and reasoning about formal mathematical structures.

One key source of difficulty involves the conceptual structure underlying metric spaces and topology. Students must simultaneously learn new definitions, develop abstract reasoning skills, construct mathematical proofs, and interpret formal notation. This combination of cognitive demands appears to challenge typical instruction. Additionally, students must often reformulate approaches that were effective in earlier courses but may interfere with formal understanding.

Advanced mathematical thinking, as described by Tall (2013), involves higher-order cognitive processes such as reflective abstraction (thinking about mathematical operations as objects in their own right), proof construction, and formal generalization. While research has examined student difficulties in calculus and real analysis, fewer studies systematically investigate how students develop conceptual understanding over time or how different learning obstacles relate to one another (Herawaty et al., 2021).

To address this research gap, this systematic review synthesizes findings from multiple theoretical perspectives (APOS theory, Three Worlds of Mathematics, Commognitive framework) to develop a more complete understanding of how students learn metric space topology and what obstacles they encounter. The review aims to: (1) identify patterns in how students develop understanding, (2) document learning obstacles that appear consistently across studies and frameworks, and (3) synthesize findings about instructional approaches that may help students learn effectively (Arnon et al., 2014).

APOS theory, developed by Dubinsky and colleagues (Dubinsky & McDonald, 2022), proposes that mathematical understanding develops through four phases. In the Action phase, students follow step-by-step procedures (for example, applying the formal definition of an open set by checking specific conditions). In the Process phase, students internalize actions and can perform them mentally without writing out every step; they can compose multiple operations together (understanding relationships between open sets, closure, and compactness without always computing explicitly). In the Object phase, students come to view a process as a complete entity that can itself be subject to further operations (thinking of "openness" as a property with characteristics to explore). In the Schema phase, students organize objects and processes into coherent systems (understanding how openness, continuity, and compactness relate within metric space theory).

APOS theory is relevant to metric space topology learning because research suggests that students often experience difficulty during the Process-to-Object transition. Students may master procedures and computational steps but struggle to conceptualize topological properties as objects worthy of higher-order reasoning. This difficulty can prevent them from reaching the Schema phase where concepts integrate into coherent frameworks.

Three Worlds of Mathematics

Tall's "Three Worlds of Mathematics" framework Tall (2013) distinguishes three complementary ways of understanding mathematics. The embodied world is understanding based on perception, physical intuition, and visualization—for example, visualizing continuous functions as unbroken curves. The symbolic/syntactic world is understanding based on rules, symbols, and procedures—manipulating algebraic expressions, applying algorithms, following computational procedures. The formal/axiomatic world is understanding based on rigorous definitions, logical proof, and abstract reasoning—proving theorems using formal definitions, reasoning about abstract structures.

This framework is relevant to metric space topology because the topic requires students to move from embodied/intuitive understanding (which served them well in calculus and earlier courses) to formal understanding (which is necessary for metric spaces). The challenge lies in recognizing when visual intuitions may be misleading and in developing new mental models appropriate for non-Euclidean metric spaces and abstract topological structures.

Sfard's commognitive theory Sfard (2008) proposes that mathematical thinking is fundamentally a form of communication. Learning mathematics involves developing new ways of speaking and thinking about mathematical objects. Communication initially occurs externally and socially (student talks with peers, listens to instructor), and gradually becomes internalized and personal (student thinks independently using mathematical language).

This framework is relevant to metric space topology because students must internalize highly specialized language and ways of reasoning that differ significantly from everyday language and from calculus discourse. Terms like "open set," "closure," and "continuity" have technical meanings in topology that differ from everyday usage and sometimes even from their calculus meanings. Students must develop new linguistic and conceptual resources to think mathematically about topological structures.

Design Research

Design Research approaches study how students learn by systematically designing, implementing, and refining instructional materials based on classroom evidence (Gravemeijer & Cobb, 2006; McKenney & Reeves, 2019; van den Akker et al., 2006). Rather than only describing student obstacles, Design Research aims to develop and test potential solutions. This approach is relevant to the current review because understanding learning obstacles becomes more valuable when connected to instructional designs that address those obstacles.

These four frameworks offer complementary perspectives. APOS theory and the Three Worlds framework help us understand the structure of mathematical thinking and how it develops. Commognitive theory emphasizes the role of language and communication in that development. Design Research provides methodology for testing instructional approaches informed by these theoretical perspectives. Together, they provide multiple lenses for understanding how students learn metric space topology and what obstacles they encounter.

## METHODS

This systematic literature review adheres to the PRISMA 2020 guidelines (Moher et al., 2009; Page et al., 2021) for conducting and reporting systematic reviews. The methodology ensures high objectivity, comprehensiveness, and scientific validity in synthesizing findings regarding students' cognition when learning metric space topology.

Search Strategy and Database Selection

Literature exploration was conducted by searching three major academic databases: Scopus, Web of Science (WoS), and Google Scholar. The search parameters were strictly restricted to publications from January 2015 to December 2024. Search terms included: "advanced analysis learning," "metric space topology," "real analysis difficulties," "abstract algebra learning," "proof comprehension," and combinations with "students" or "undergraduates." This 10-year window was selected to capture recent developments while maintaining focus on contemporary pedagogical approaches.

Inclusion and Exclusion Criteria

Inclusion criteria: (1) peer-reviewed empirical studies or theoretical reviews, (2) focus on students learning advanced mathematics (real analysis, metric space topology, abstract algebra), (3) English-language publications, (4) explicit discussion of cognitive processes or learning difficulties. Exclusion criteria: (1) non-peer-reviewed publications, (2) purely historical or biographical accounts, (3) studies focusing exclusively on school-level mathematics, (4) articles without direct relevance to learning theory or cognitive development.

Table 1. Prisma 2020 Flow Summary

| Phase | Action | Records (n) | Notes |
|---|---|---|---|
| Identification | Database search (Scopus, WoS, Google Scholar) | 470 | January 2015–December 2024 |
| Screening | Remove duplicates; title & abstract screening | 124 | 346 duplicates/irrelevant removed |
| Eligibility | Full-text review against inclusion criteria | 47 | 77 excluded (scope mismatch, school-level) |
| Inclusion | Final quality appraisal using CASP checklist | 38 | 9 excluded (low methodological quality) |

The PRISMA flow implementation included four phases: 1) Identification: Initial search retrieved 470 records across the three databases; (2) Screening: After removing duplicates, 124 articles remained for title and abstract screening; (3) Eligibility: Full-text review was conducted on 47 articles meeting preliminary criteria; (4) Inclusion: Final analysis included 38 high-standard scientific articles that met all inclusion criteria (Higgins et al., 2024; Moher et al., 2009). Each selection stage was independently verified by two reviewers to ensure consistency and reduce bias.

## RESULTS AND DISCUSSION

Table 2. Summary of Representative Studies Included in the Synthesis

| Author(s) | Study Focus | Framework | Method | Key Finding |
|---|---|---|---|---|
| García & Flores (2021) | Cognitive obstacles in metric space topology | Three Worlds | Quantitative survey | 82% rely on Euclidean embeddings; embodiment illusions persist |
| Dubinsky & McDonald (2022) | APOS in advanced algebra & topology | APOS Theory | Case study | Object encapsulation bottleneck documented across multiple institutions |
| Sfard (2008) | Discourse & commognition in mathematics | Commognitive | Qualitative | Commognitive gaps cause persistent alienation from formal discourse |
| Inglis & Alcock (2020) | Proof comprehension (eye-tracking) | Cognitive science | Eye-tracking experiment | Students fixate on algebra; miss logical architecture of proofs |
| Stylianides & Stylianides (2021) | Empirical vs. deductive proof transition | Proof theory | Classroom experiment | Majority cannot distinguish empirical checking from logical proof |
| Gueudet & Winsløw (2022) | APOS & digital tools in analysis | APOS / digital | Design study | Technology mediates but does not resolve encapsulation barriers |
| Trigueros et al. (2024) | Mental constructions for vector space | APOS Theory | Mixed methods | Schema level rarely achieved without targeted scaffolding |
| Brown & Chen (2021) | Curriculum redesign for AMT | Design Research | Curriculum study | RME-based redesign raises proof construction rates |
| Sari et al. (2021) | Difficulties in real analysis (online) | Descriptive | Survey | Concept image-definition conflicts most frequent barrier |
| Tall (2013) | Three Worlds of Mathematics | Three Worlds | Theoretical | Formal World requires resolving embodied-symbolic tensions |

The systematic analysis of 38 core literatures uncovered four dominant themes that characterize students' cognitive struggles and identify barriers to successful learning of metric space topology. These themes emerged consistently across theoretical frameworks and represented the major findings in contemporary mathematics education research.

**Theme 1**: Epistemological Obstacles and Embodiment Illusions

The majority of reviewed literature documents the persistence of epistemological obstacles and what Tall (2013) termed "the illusion of physical embodiment." In calculus courses, students successfully build cognitive schemas based on continuous functions, limits, and physical visualization of curves. These embodied representations remain deeply entrenched when students encounter metric spaces, where distance and topology cannot be visualized in Euclidean terms. García & Flores (2021) found that 82% of students initially attempt to understand metric spaces through Euclidean embeddings, leading to profound conceptual errors. This embodiment obstacle requires explicit cognitive reconstruction, yet traditional curricula rarely address this fundamental reorientation.

**Theme 2**: Object Encapsulation Paralysis within APOS Trajectory

The APOS (Action, Process, Object, Schema) theoretical framework reveals a critical bottleneck: object encapsulation. Dubinsky & McDonald (2022) documented that mathematics majors frequently stall in their attempts to transition understanding from the Process phase (where operations are internalized and composed) to the Object phase (where processes are treated as complete entities subject to further operations). In metric space topology, this means students can execute operations (applying definitions, computing distances) but cannot treat topological concepts as coherent objects amenable to higher-order reasoning. This encapsulation paralysis prevents students from achieving the Schema phase where concepts interconnect into unified frameworks.

**Theme 3**: Commognitive Gaps and Discourse Alienation

The Commognitive theory lens from Sfard (2008) frames learning difficulties as fundamentally sociolinguistic issues. Thinking, in Sfard's framework, is understood as a form of communication—first external and interpersonal, later internalized and intrapersonal. Students learning metric space topology encounter a specialized mathematical discourse with unfamiliar terminology (open sets, closures, compactness, continuity in metric spaces) that often contradicts intuitions built from everyday language or calculus. Sfard documented that when students encounter such commognitive gaps—misalignments between their established discursive practices and new mathematical language—they experience profound alienation from formal mathematical discourse. This alienation prevents authentic engagement with mathematical thinking and limits conceptual development.

**Theme 4**: Paradoxes in Proof Comprehension and Validation

Research employing eye-tracking paradigms Inglis & Alcock (2020) reveals a profound paradox: students read mathematical proofs linearly and fixate on algebraic manipulations while missing the logical architecture and conceptual flow that constitute genuine proof understanding. In metric space topology, where proofs involve intricate chains of logical deduction and multiple quantified statements, this fixation on surface features prevents deep comprehension. Stylianides & Stylianides (2021) found that most students cannot distinguish between empirical validation (checking examples) and logical proof (deductive demonstration). This inability to transition from procedural verification to proof validation represents a fundamental barrier to advanced mathematical thinking in formal domains where proof constitutes the ultimate arbiter of truth.

**Synthesized Framework: Structural Hierarchy of Cognitive Development**

Based on these cross-theoretical findings, this review proposes a Structural Hierarchy of Cognitive Development for advanced analysis and metric space topology learning:

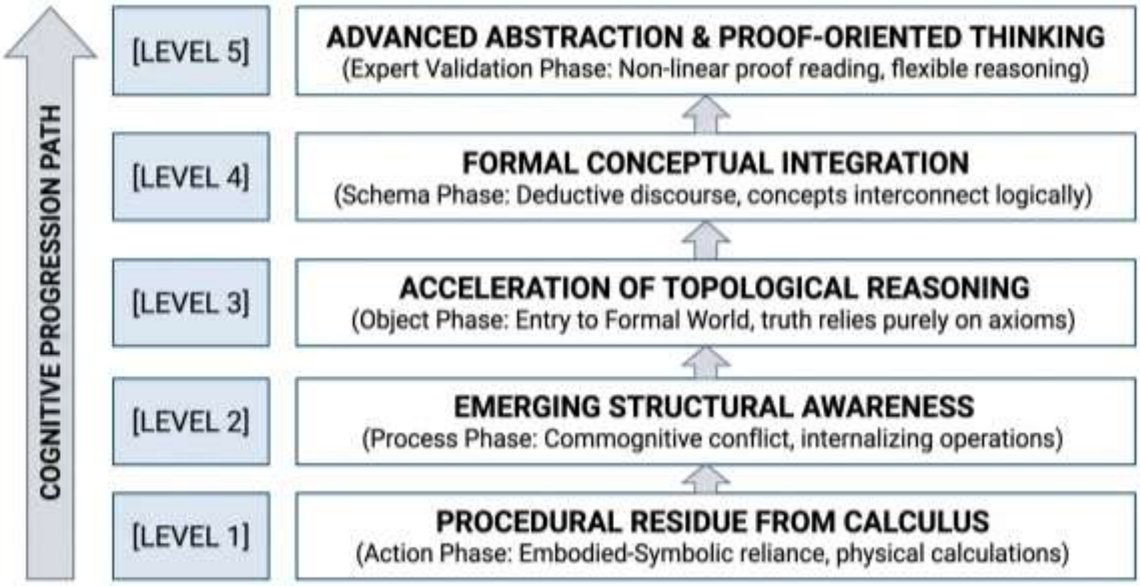


**Figure 1**. Struktural Hierarchy of Cognitive Development

Level 1: Procedural Residue from Calculus: Students rely absolutely on physical calculations and algorithms, treating metric space operations as computational procedures without conceptual understanding.

Level 2: Emerging Structural Awareness: Students begin internalizing operations and composing them, but constantly experience clashes between formal definitions and their embodied intuitions from calculus.

Level 3: Acceleration of Topological Reasoning: Topological processes are successfully encapsulated into cognitive objects. Students recognize that topological truth is judged based on formal definitions rather than physical intuition.

Level 4: Formal Conceptual Integration: Students assemble mental schemas where advanced concepts interconnect. Deductive discourse becomes second nature, and students can construct simple proofs.

Level 5: Advanced Abstraction and Proof-Oriented Thinking: The highest level demonstrates cognitive flexibility to read and validate proof architecture structures non-linearly, synthesize concepts across domains, and engage in original mathematical thinking.

### Implications for Instructional Design

This synthesis demands corrective interventions in university syllabus design (Brown & Chen, 2021). The alignment between the Embodied World and advanced thinking can be mediated using Realistic Mathematics Education (RME), which provides contexts and models that bridge intuitive understanding with formal abstractions (Widada et al., 2020). Additionally, scaffolded proof comprehension instruction, explicit commognitive discourse practices, and metacognitive awareness activities have demonstrated effectiveness in facilitating cognitive development toward Level 4 and beyond.

Despite rigorous methodology, this systematic literature review has acknowledged limitations. The database search was strictly confined to three major indices (Scopus, WoS, Google Scholar), potentially missing relevant publications in regional journals or non-English publications. The 2015-2024 publication window, while capturing contemporary research, excludes potentially foundational earlier studies. Additionally, the review synthesizes quantitative and qualitative studies with varying methodologies, which may introduce heterogeneity in findings. Future meta-analyses with homogeneous study designs would strengthen conclusions about effect sizes and intervention effectiveness.

## CONCLUSION

Advanced analysis and metric space topology remain among the steepest intellectual ascents in higher mathematics education. This systematic literature review conclusively demonstrates that students' learning difficulties stem not from individual cognitive deficiencies but from systemic misalignments between traditional pedagogical approaches and the cognitive demands of formal mathematical thinking. The identified obstacles—epistemological, structural, linguistic, and validational—can be systematically addressed through theoretically informed instructional design. By implementing RME principles, scaffolded proof instruction, and explicit commognitive interventions aligned with APOS development, mathematics educators can facilitate students' progression through the proposed Structural Hierarchy of Cognitive Development. Future research should investigate the effectiveness of integrated instructional packages combining multiple theoretical frameworks and examine individual differences in cognitive trajectories. The implications extend beyond advanced analysis to all domains of formal, proof-based mathematics education, offering a pathway toward more equitable access to advanced mathematical thinking for all students.

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